Indian Statistical Institute

Semestral Examination

Differential Geometry II - MMath II

Max. Marks : 60 Tome : 3 hours

(1) (a) Define the term : *Riemannian metric*. Given $A, B \in M_n(\mathbb{R})$, $g(A, B) = \mathrm{tr}(A^t B)$ is a metric on $M_n(\mathbb{R})$. Justify. [3]

(b) Let (M, g) be a Riemannian manifold, $\sigma : I \longrightarrow M$ a smooth curve and d_g the associated metric on M. Define the length $\ell(\sigma)$ of σ. If $M = M_n(\mathbb{R})$ and g the metric in question 1 (a) above, show that $d_g(0, Id) \leq \sqrt{n}$. Here 0 is the zero matrix and Id denotes the identity matrix. [1+6]

(c) Define the term : *isometry*. On $O(n, \mathbb{R})$ consider the metric g induced by the metric in question 1 (a). Show that left multiplication by elements of $O(n, \mathbb{R})$ are isometries. [10]

(2) (a) State the fundamental theorem of Riemannian geometry. [2]

(b) Let $\mathbb{H}$ denote the upper half plane with the Poincaré metric g. Describe the Riemannian connection on $(\mathbb{H}, g)$. [12]

(c) Show that the map $f : (\mathbb{H}, g) \longrightarrow (\mathbb{H}, g)$ defined by $(x, y) \mapsto (x + a, y)$, $a \in \mathbb{R}$ is an isometry. Is the map $h : (\mathbb{H}, g) \longrightarrow (\mathbb{H}, g)$ defined by $(x, y) \mapsto (x, y + b)$, $b \in \mathbb{R}$ an isometry? [6]

(3) (a) Discuss the notion of parallel transport on a Riemannian manifold. Show by an example that parallel transport depends on the path. [10]

(b) Define the terms : *geodesic, geodesically complete*. Describe the geodesics on the sphere. Conclude that the sphere is geodesically complete. Give an example of a Riemannian manifold that is not geodesically complete. [10]